PARAMETRIC METHOD IN THE THEORY OF A PERIODIC BOUNDARY LAYER WITH LARGE STROUHAL NUMBERS

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A universal equation for the Linn problem on a periodic boundary layer is obtained and solved numerically with certain values of the parameters.

The method presented in [1, 2], as a simple analysis shows, can be used in the calculation of a periodic laminar boundary layer only with small Strouhal numbers. The use of a parametric method for the case of large Strouhal numbers is discussed in the present article. The solution of the problem of an oscillating laminar boundary layer in an incompressible liquid proposed by Linn [3] is used for this purpose.

Let us recall the main propositions of Linn's theory. The velocity $U(x, t)$ at the outer limit of a boundary layer is considered as the sum of its average value $\bar{U}(x)$ over the period and of some periodic contribution $U_1(x, t)$. The velocity components and pressure in the boundary layer are assigned in such a form. The operation of averaging over the period and a number of simple transformations lead to equations for the average motion in the boundary layer in the form

$$\left.\begin{aligned} \bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y} &= \bar{U}\frac{d\bar{U}}{dx}+\nu\frac{\partial^2\bar{u}}{\partial y^2}+R(x, y),\\ \frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y} &= 0. \end{aligned}\right\} \tag{1}$$

The equations obtained differ from the usual equations of a steady laminar boundary layer by the presence of the term $R(x, y)$, which can be considered as an additional volumetric force of an inertial nature. Linn calculates the quantity $R(x, y)$ using equations which he obtains for the oscillatory components of the motion. In this case, when the Strouhal number is large and the velocity at the outer limit of the boundary layer is assigned by a harmonic law in the form $U(x, t) = \bar{U}(x) + W(x)\sin\omega t$, we will have

$$R(x, y) = \frac{W}{2}\cdot\frac{dW}{dx}R_1(\xi), \tag{2}$$

$$R_1(\xi) = \exp(-\xi)\left[(2+\xi)\cos\xi-(1-\xi)\sin\xi-\exp(-\xi)\right]. \tag{3}$$

Here $\xi = y/\delta_0$, $\delta_0 = \sqrt{2\nu/\omega}$. The quantity δ_0, which can be called the thickness of the oscillatory boundary layer, has the same order with respect to the Reynolds number of the stream as does the thickness of the steady boundary layer. The solution of Linn's system of equations (1)-(3) naturally depends on the form of the functions $\bar{U}(x)$ and $W(x)$ and the oscillation frequency ω. Let us transform this system so that the indicated quantities do not enter into it explicitly, in other words, let us reduce the Linn equations to a universal form. We introduce a stream function, representing it in the form of a sum of the stream functions for the average and oscillatory motions so that

$$\psi(x, y, t) = \bar{\psi}(x, y) + \psi_1(x, y, t); \quad \bar{\psi}_1 = 0.$$

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Changing to the new variables

$$x = x, \quad \eta = \frac{By}{h(x)}, \quad \varphi(x, \eta) = \frac{B\bar{\psi}(x, y)}{\bar{U}(x)h(x)}, \tag{4}$$

where h(x) is the characteristic transverse linear scale in the boundary layer, taken for the average motion, and B is a normalizing constant, with allowance for (2) and (3) we can transform Eq. (1) to the form ($z = h^2/\nu$)

$$B^2 \frac{\partial^3\varphi}{\partial\eta^3} + \left(\bar{U}'z + \frac{\bar{U}}{2} z'\right) \varphi \frac{\partial^2\varphi}{\partial\eta^2} + \bar{U} z\left(\frac{\partial\varphi}{\partial x} \cdot \frac{\partial^2\varphi}{\partial\eta^2} - \frac{\partial\varphi}{\partial\eta} \cdot \frac{\partial^2\varphi}{\partial x\partial\eta}\right) - \bar{U}'z\left[\left(\frac{\partial\varphi}{\partial\eta}\right)^2 - 1\right] + \frac{1}{2} \frac{W}{\bar{U}} W'zR_1(\xi) = 0. \tag{5}$$

Here

$$\xi = \frac{y}{\delta_0} = \frac{h}{B\delta_0}\eta = \frac{1}{B}\sqrt{\frac{\omega}{2} z}\,\eta; \tag{6}$$

the total derivative with respect to x is denoted by a prime.

We can assign the multiparametric family of average velocity profiles and, accordingly, the stream functions in the boundary layer in the form

$$\frac{\bar{u}}{\bar{U}} = \frac{\bar{u}}{\bar{U}}(\eta, f_k, q_n, f_\omega); \quad \bar{\psi} = \frac{\bar{U}h}{B}\varphi(\eta, f_k, q_n, f_\omega);$$

$$k = 1, 2, 3, \ldots, \quad n = 0, 1, 2, \ldots .$$

The system of parameters is determined by the following equations:

$$f_k = \bar{U}^{k-1} \frac{d^k\bar{U}}{dx^k} z^k; \quad q_n = \frac{W^n}{\bar{U}} \cdot \frac{d^nW}{dx^n} z^n; \quad f_\omega = \sqrt{\frac{\omega}{2} z}. \tag{7}$$

It is assumed that the functions $\bar{U}(x)$ and W(x) are analytical. The infinite sequence of parameters f_k is analogous to Loitsyanskii's series of parameters which he introduced in solving the problem of a steady boundary layer [4]. The system of parameters q_n reflects the effect of the oscillatory motion with an amplitude W(x) in the outer stream on the average motion in the boundary layer. The first of these parameters $q_0 = W/\bar{U}$ represents the relative amplitude of the velocity oscillations; the next parameter $q_1 = (W/\bar{U})W'z$, in addition to the relative amplitude, also includes the first derivative with respect to the longitudinal coordinate of the amplitude of the oscillations of the outer stream. A positive value of q_1 corresponds to oscillations which increase along the surface over which the flow occurs, while a negative value corresponds to damped oscillations. The parameter $f_\omega = \sqrt{(\omega/2)\cdot z}$ is an independent parameter which reflects the effect of the oscillation frequency of the outer stream on the average motion in the boundary layer. The previous history of the motion in the boundary layer is taken into account through the quantity z. With arbitrary functions $\bar{U}(x)$ and W(x) all the parameters introduced are independent and are henceforth considered as independent variables.

Assuming that the function $\varphi(\eta, f_k, q_n, f_\omega)$ of an infinite set of arguments is analytical, we can write

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial f_\omega} \cdot \frac{df_\omega}{dx} + \sum_{k=1}^{\infty} \frac{\partial\varphi}{\partial f_k} \cdot \frac{df_k}{dx} + \sum_{n=0}^{\infty} \frac{\partial\varphi}{\partial q_n} \cdot \frac{dq_n}{dx}, \tag{8}$$

with the following equations for the derivatives of the parameters being easy to obtain by differentiation of Eqs. (7):

$$\frac{df_k}{dx} = \frac{\theta_k^{(f)}}{\bar{U}z}, \text{where } \theta_k^{(f)} = (k-1)f_1f_k + kFf_k + f_{k+1};$$

$$\frac{dq_n}{dx} = \frac{\theta_n^{(q)}}{\bar{U}z}, \text{where } \theta_n^{(q)} = n\frac{q_1}{q_0^2}q_n - f_1q_n + \frac{q_{n+1}}{q_0}; \tag{9}$$

Here we introduce the designation

$$\frac{df_\omega}{dx} = \frac{\theta^{(\omega)}}{\bar{U} z}, \text{where } \theta^{(\omega)} = \frac{f_\omega F}{2}.$$

$$F = \bar{U} z'. \tag{10}$$

Using Eqs. (6)-(10) we can transform Eq. (5) to the following form:

$$B^2 \frac{\partial^3 \varphi}{\partial \eta^3} + \frac{F + 2f_1}{2} \varphi \frac{\partial^2 \varphi}{\partial \eta^2} + f_1 \left[1 - \left(\frac{\partial \varphi}{\partial \eta}\right)^2\right] + \frac{q_1}{2} R_1 \left(\frac{f_\omega \eta}{B}\right) +$$
$$+ \frac{\partial^2 \varphi}{\partial \eta^2} \left(\frac{\partial \varphi}{\partial f_\omega} \theta^{(\omega)} + \sum_{k=1}^{\infty} \frac{\partial \varphi}{\partial f_k} \theta_k^{(f)} + \sum_{n=0}^{\infty} \frac{\partial \varphi}{\partial q_n} \theta_n^{(q)}\right) -$$
$$- \frac{\partial \varphi}{\partial \eta} \left(\frac{\partial^2 \varphi}{\partial \eta \partial f_\omega} \theta^{(\omega)} + \sum_{k=1}^{\infty} \frac{\partial^2 \varphi}{\partial \eta \partial f_k} \theta_k^{(f)} + \sum_{n=0}^{\infty} \frac{\partial^2 \varphi}{\partial \eta \partial q_n} \theta_n^{(q)} \right) = 0. \tag{11}$$

In order for the equation obtained to be universal, it is necessary that the function $F = \bar{U}z'$ entering into it be expressed through the parameters and functions which depend only on the parameters. This can be done with the help of the integral impulse equation, which is easily derived for the average motion on the basis of Eqs. (1)-(3). Simple transformations of these equations and integration across the boundary layer from zero to infinity lead to the following integral equation:

$$\frac{d}{dx} (\bar{U}^2 \delta^{**}) + \bar{U} \bar{U}' \delta^* = \frac{\tau_w}{\rho} - \frac{1}{4} \delta_0 W W'. \tag{12}$$

Here

$$\delta^* = \int_0^\infty \left(1 - \frac{\bar{u}}{\bar{U}}\right) dy; \quad \delta^{**} = \int_0^\infty \frac{\bar{u}}{\bar{U}} \left(1 - \frac{\bar{u}}{\bar{U}}\right) dy;$$

$$\tau_w = \mu \left. \frac{\partial \bar{u}}{\partial y} \right|_{y=0}. \tag{13}$$

Using the variables (4), one can write

$$\delta^* = hH^*, \; \delta^{**} = hH^{**}, \; \tau_w = \frac{\mu \bar{U}}{h} \zeta, \tag{14}$$

where the quantities

$$H^* = \frac{1}{B} \int_0^\infty \left(1 - \frac{\partial \varphi}{\partial \eta}\right) d\eta, \quad H^{**} = \frac{1}{B} \int_0^\infty \frac{\partial \varphi}{\partial \eta} \left(1 - \frac{\partial \varphi}{\partial \eta}\right) d\eta,$$

$$\zeta = B \left. \frac{\partial^2 \varphi}{\partial \eta^2} \right|_{\eta=0}$$

depend only on the parameters. Using Eqs. (8) and (9), one can change from the longitudinal coordinate x to the spatial parameters in Eq. (12). If one chooses the thickness of impulse loss as the scale of the transverse coordinate in the boundary layer, i.e., $h(x) = \delta^{**}(x)$, then Eq. (12) takes the form

$$F = 2 \left[\zeta - f_1 (2 + H) - \frac{1}{4} \frac{q_1}{f_w}\right], \tag{15}$$

where, in accordance with (14), we will have

$$H = H^* = \frac{\delta^*}{\delta^{**}}, \quad \zeta = \frac{\tau_w \delta^{**}}{\mu \bar{U}}.$$

It follows from (15) that the functional F is composed of quantities which depend only on the parameters, and therefore the substitution of Eq. (15) into Eq. (11) makes the latter universal, i.e., not explicitly containing the velocity at the outer limit of the boundary layer (neither the velocity of the average motion nor the characteristics of the oscillatory component). The universal equation (11) is integrated once and for all with the following boundary conditions:

$$\varphi = \frac{\partial \varphi}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0, \quad \frac{\partial \varphi}{\partial \eta} \to 1 \quad \text{as} \quad \eta \to \infty, \tag{16}$$

$$\varphi = \varphi_0(\eta) \quad \text{at} \quad f_k = q_n = f_\omega = 0, \quad k = 1, 2, 3, \ldots, \quad n = 0, 1, 2, \ldots,$$

where $\varphi_0(\eta)$ is the Blasius solution for the steady boundary layer at a plate. In this case the normalizing constant is B = 0.47.

The infinite number of variables in Eq. (11) forces one to be confined to "fragments" of this equation with a finite number of parameters in the numerical integration. Let us consider an approximation which can be called local with respect to three parameters. In this approximation the derivatives with respect to all the parameters are discarded and only the parameters f_1, q_1, and f_ω are retained. We note that the parameter q_0 is not taken into account, since it enters only into terms containing derivatives with respect to the parameters. Equation (11) in the local approximation with respect to three parameters has the form [the function R_1 is disclosed through Eq. (3)]

$$B^2 \frac{d^3\varphi}{d\eta^3} + \frac{F + 2f_1}{2} \varphi \frac{d^2\varphi}{d\eta^2} + f_1 \left[1 - \left(\frac{d\varphi}{d\eta}\right)^2\right] +$$
$$+ \frac{q_1}{2} \exp\left(-\frac{f_\omega}{B}\eta\right) \left[\left(2 + \frac{f_\omega}{B}\eta\right) \cos \frac{f_\omega}{B}\eta -\right.$$
$$\left. - \left(1 - \frac{f_\omega}{B}\eta\right) \sin \frac{f_\omega}{B}\eta - \exp\left(-\frac{f_\omega}{B}\eta\right)\right] = 0, \tag{17}$$
$$\varphi = \frac{d\varphi}{d\eta} = 0 \quad \text{at} \quad \eta = 0; \qquad \frac{d\varphi}{d\eta} \to 1 \quad \text{as} \quad \eta \to \infty.$$

The functional F entering into this equation is calculated from Eq. (15), with $\zeta = \zeta(f_1, q_1, f_\omega)$ and $H = H(f_1, q_1, f_\omega)$.

Equation (17) was integrated on a computer by the trial-run method with iterations. The velocity profiles and the characteristic functions ζ, F, and H in dependence on the parameters f_1, q_1, and f_ω are obtained as a result. Some of the calculated curves are presented in Fig. 1. It is seen from the graphs that the amplitude parameter $q_1 = (W/\bar{U})W'z$ has a marked effect on the magnitude of the average reduced friction and on the average position of the point of separation. With an increase in the positive value of this parameter the average friction in the boundary layer increases, while the point of separation is displaced in the direction of a larger negative value of the parameter f_1. Hence it follows that the presence of oscillations with an amplitude which increases along the coordinate leads to protraction of the separation in the diffusor. Damping oscillations ($q_1 < 0$ because $W' < 0$), conversely, stimulate the separation. When $q_1 = 0$, which corresponds to $W = 0$ or $W' = 0$, the average motion in the boundary layer coincides with the steady motion. Thus, as also follows from Linn's work [3], with large Strouhal numbers the oscillations of the outer stream with a constant amplitude along the boundary layer do not affect the characteristics of the boundary layer; in this case the value of the relative amplitude which is possible within the framework of the theory of a laminar boundary layer is not important. It is interesting to note that the latter arguments are valid only in the case of the local approximation discussed. But if one allows for the derivative with respect to the parameter q_0 in Eq. (11), then the quantity $\theta_0^{(q)} = (q_1/q_0) - f_1 q_0$ will include in explicit form the parameter q_0, which is equal to the dimensionless amplitude. Thus, the parametric method also allows one to estimate the effect of the amplitude of the oscillations on the characteristics of the boundary layer in the case when $W' = 0$, i.e., $q_1 = 0$.

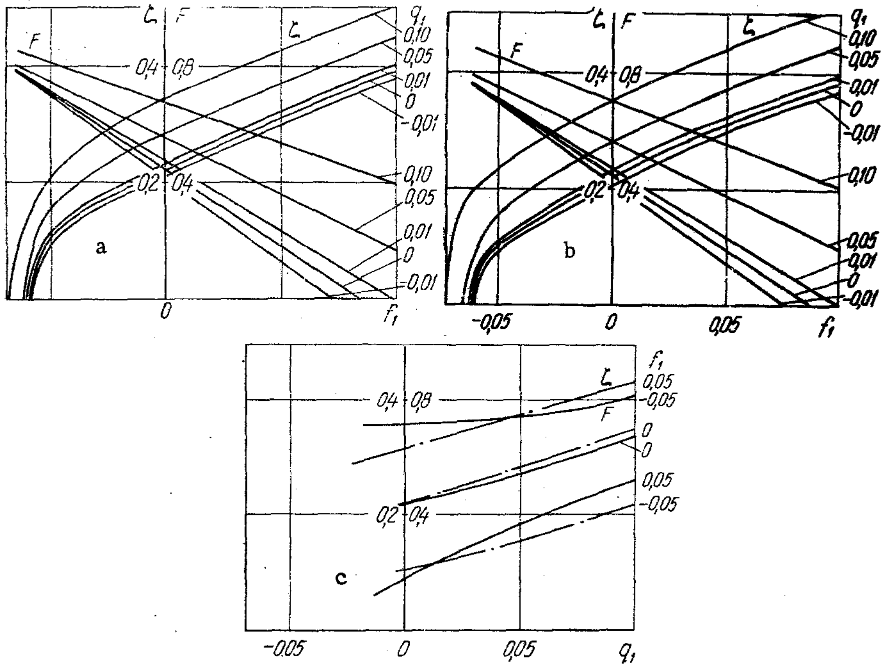

Fig. 1. Dependence of reduced coefficient of friction and characteristic function: a) on the parameter f_1 with q_1 = const and $f_\omega = 2$; b) on the parameter f_1 with q_1 = const and $f_\omega = 1$; c) on the parameter q_1 with f_1 = const and $f_\omega = 2$.

As for the influence of the frequency parameter $f_\omega = \sqrt{(\omega/2)z}$ on the characteristics of the boundary layer, it is unimportant in the range analyzed. As follows from a comparison (Fig. 1a, b), however, with an increase in f_ω the curves of reduced friction for different values of q_1 differ from the corresponding curve for steady motion ($q_1 = 0$), i.e., the influence of the amplitude parameter increases somewhat with an increase in the oscillation frequency. Therefore, when $q_1 > 0$ some increase in friction in the boundary layer and protraction of the separation are observed with an increase in the oscillation frequency of the outer stream. An increase in frequency leads to the opposite effect when $q_1 < 0$.

As seen from Fig. 1, the graphs of $F(f_1)$ and $F(q_1)$ are close to straight lines, although the marked difference in slopes admits of a rough linear approximation of the function F only for small values of the parameters. Neglecting the influence of the parameters f_ω on the function F in their analyzed range, we will have

$$F = 0.44 - 5.35 f_1 + 2.10 q_1. \tag{18}$$

A distinctive aspect of the effect of oscillations on the average stream is the appearance of inflection points in the velocity profiles, which has been noted by a number of investigators [3, 5]. In the integration of the universal equation in our case similar results are obtained with positive values of the parameter q_1.

For the solution of a concrete problem when the velocity distribution in the outer stream is known it is necessary to establish the relation between the parameters and the longitudinal coordinate x. For this purpose one must turn to the impulse equation, which is written in the following form:

$$\frac{df_1}{dx} = \frac{\bar{U}'}{\bar{U}} F + \frac{\bar{U}''}{\bar{U}'} f_1; \tag{19}$$

the form of the function $F = F(f_1, q_1, f_\omega)$ should be known as a result of the solution of the universal equation. Using the obvious relations

$$q_1 = \frac{W}{\bar{U}} \cdot \frac{W'}{\bar{U}'} f_1, \quad f_\omega = \sqrt{\frac{\omega}{2} \cdot \frac{f_1}{\bar{U}'}} \tag{20}$$

the function F is transformed in such a way that it remains an explicit dependence only of the parameter f_1 and the characteristics of the outer stream. When (18) is used the computation is carried out only for the first equation of (20). Knowing the velocity distribution in the outer stream, one can solve Eq. (19) and consequently determine the dependence $f_1(x)$. Then the functions $q_1(x)$ and $f_\omega(x)$ are established with the help of Eqs. (20). From the values of the parameters known for each given cross section of the boundary layer, obtained once and for all in the integration of the universal equation using graphs or tables, one can find the average reduced friction and the average velocity field in a given cross section of the boundary layer. It should be noted that in the general case Eq. (19) is nonlinear and requires approximate integration.

To estimate the accuracy of the proposed method we calculated the average reduced friction at the front critical point of a body (F = 0) with assignment of the velocity at the outer limit of the boundary layer in the form $U(x, t) = Ax(1 + \varepsilon \sin \omega t)$. With $q_1 = 0.01$, $f_\omega = 2$, and $f_1 = 0.097$ we will have $\zeta = 0.39$ (see Fig. 1a). An exact solution [5] under the same conditions gives $\zeta = 0.346$. The disagreement with the value obtained above is 12%. Refinement of the solution of the problem can be obtained by the integration of a "fragment" of the universal equation with the retention of derivatives with respect to the parameters.

NOTATION

x,y, longitudinal and transverse coordinates in boundary layer; t, time; η, dimensionless transverse coordinate; U, velocity at outer limit of boundary layer; u,v, projections of velocity in boundary layer on x and y axes, respectively; μ,ν, coefficients of dynamic and kinematic viscosity, respectively; W(x), amplitude of periodic component of velocity at outer limit of boundary layer; ω, frequency of velocity oscillations; δ_0, thickness of oscillatory boundary layer; ψ, stream function; φ, dimensionless stream function of average motion of liquid in boundary layer; h(x), scale of transverse coordinate in boundary layer; $z = h^2/\nu$; f_k, q_n, f_ω, dimensionless parameters; B, normalizing constant; δ^*, displacement thickness; δ^{**}, thickness of impulse loss; τ_w, surface friction stress; F, H, H*, H**, characteristic functions; ζ, reduced coefficient of surface friction; A, constant coefficient.

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